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ESTIMATION OF LAUNCH VEHICLE PERFORMANCE

PARAMETERS FROM AN ORBITING SENSOR

THESIS

AFIT/GA/AA/81D-9 Gregory D. Miller
Captain USAF

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ESTIMATION OF LAUNCH VEHICLE PERFORMANCE PARAMETERS FROM AN ORBITING SENSOR

THESIS

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

Air University

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Master of Science

by

Gregory D. Miller, B.S.

Captain USAF

Graduate Astronautical Engineering

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Preface

This report will hopefully help in paving the way toward a useful algorithm for the estimation of launch vehicle parameters. Although no such algorithm is contained herein, I feel that I've identified some major problems and perhaps have recommended reasonable approaches to solutions to those problems.

This work could not have become a reality without the typing efforts of Ladonna Stitzel. Her help is deeply appreciated. I'd also like to express special thanks to my advisor, Dr. Bill Wiesel. A thesis effort which doesn't meet personal goals can be frustrating, but his guidance and inspiration made this a fruitful learning experience.

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Abstract

A seven-state inverse covariance (Bayes) filter was implemented to determine performance parameters of a launch vehicle. Data measurements were restricted to azimuth and elevation readings, typical of data from an infrared sensor in geosynchronous orbit. Results of this study indicate that the magnitude of constant acceleration, assumed to act in the direction of velocity, can be estimated using a seven-state filter (3 states each for position and velocity, and a seventh state for acceleration). The system is unobservable for short arcs of data if only one observer is available. The addition of a second observer can allow the system to be observed. An ad hoc fading - memory technique, in which confidence in the seventh state estimate was decreased, proved unsuccessful in estimating variable acceleration of a launch vehicle. Further attempts at estimating variable acceleration with an eight-state filter (3 states each for position and velocity, and seventh and eighth states involving engine exit velocity and propellent mass flow rate) were unsuccessful.

I Introduction

With the increasing number of reconnaissance and other data-gathering satellites, a growing source of observation data is becoming available which can be used to supplement or replace ground-based or aerial data for various requirements. In some instances, this space capability allows for the gathering of data previously unavailable from the ground or air. Measurements of elevation and azimuth from satellite infrared sensors fall into this category. The use of these data in estimating parameters of ballistic missiles and reentry vehicles has therefore been limited.

Studies have been conducted in the past which concentrate on the estimation of position, velocity, and performance parameters of maneuvering reentry vehicles. The filters developed in these studies have used ground-based tracking radars as their primary measurement source. Typical data, then, has included azimuth, elevation, and range.

This paper presents the development of an inverse covariance (Bayes) filter updated by satellite data. The satellite data consist of elevation and azimuth angles (no range) as detected from an orbiting infrared sensor. These data would be taken during the ascent of rockets or ballistic missiles. Of primary interest in this development was the ability of the filter to estimate accelerations of the target vehicle in addition to its position and velocity.

Assumptions:

(1) The data satellites were assumed to be in geosynchronous equatorial orbits.

(2) Accuracy of the infrared sensor was considered the same for determining both elevation and azimuth angles.

(3) The first filter was assumed to have seven states; x, y, z components of position and velocity, and an acceleration term resulting from thrust. This model was thought to be general enough such that it might also be used to estimate deceleration of a reentry vehicle due to drag.

(4) The acceleration term was assumed to act along the velocity direction vector in order to keep the model simple.

Chapter II of this paper presents the derivation of the dynamics equations for the filter. This is followed in Chapter III by the development of observation relationships between the heat emitting ballistic missile or reentry vehicle (subsequently referred to as the target vehicle) and the satellite-based infrared sensor. In Chapter IV the filter equations are derived. In addition, Chapter IV also contains a discussion of the problems encountered during implementation of the seven-state filter. A follow-on filter with eight states is introduced and briefly analyzed in Chapter V. Chapter VI contains conclusions resulting from the development work and presents recommendations for further study.

II Derivation of Problem Dynamics

The equations of motion for the target vehicle were derived from the general two-body equation:

$$\ddot{\bar{r}} + \frac{\mu}{r^3}\bar{r} = 0 \qquad (2\text{-}1)$$

where $\mu = GM$

$$r = x\hat{i} + y\hat{j} + z\hat{k} \qquad (2\text{-}2)$$

The state model for the filter initially had seven states; three for position, three for velocity, and one for acceleration due to thrust or drag.

The state vector, $\bar{x}$, therefore was:

$$\bar{x} = \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \\ a \end{bmatrix} \qquad (2\text{-}3)$$

The state vector propagates in time according to the vector differential equation:

$$\dot{\bar{x}} = \bar{F}(\bar{x}(t),t) \qquad (2\text{-}4)$$

In $\bar{x}$, the three position states denote position components in the x-, y-, and z- directions in a coordinate frame with its origin at the earth's center and the z-direction being north. The rates of change of these states were given by their respective velocities:

$$\dot{x} = v_x \tag{2-5a}$$

$$\dot{y} = v_y \tag{2-5b}$$

$$\dot{x} = v_z \tag{2-5c}$$

The time rates of change of the three velocity states were derived from the two-body equation and the assumed representation for acceleration due to thrust (or similarly, deceleration due to drag). The acceleration will be discussed further, but was basically modeled as a vector acting in the direction of velocity and having constant magnitude over time increments between observations. Therefore acceleration was given by:

$$\bar{a} = a \frac{\bar{v}}{|\bar{v}|} \tag{2-6}$$

Adding this to the two-body equation,

$$\ddot{\bar{r}} = - \frac{\mu}{r^3} \bar{r} + a \frac{\bar{v}}{|\bar{v}|} \tag{2-7}$$

yields equations for the time rates of change of the velocity states:

$$\dot{v}_x = - \frac{\mu}{r^3} x + a \frac{v_x}{|\bar{v}|} \tag{2-8a}$$

$$\dot{v}_y = - \frac{\mu}{r^3} y + a \frac{v_y}{|\bar{v}|} \tag{2-8b}$$

$$\dot{v}_z = - \frac{\mu}{r^3} z + a \frac{v_z}{|\bar{v}|} \tag{2-8c}$$

As stated previously, the seventh state, acceleration, was modelled as constant over time intervals between observations. It was assumed to act always in the direction of the velocity. The significance of this implementation was that this model could be used to represent acceleration during ascent (a positive value) or, with a change of sign, could represent deceleration of a reentry vehicle descending through the atmosphere. This implementation would allow the same filter structure to be used from lift-off to reentry vehicle impact.

With acceleration constant between measurement updates, its time rate of change was given by:

$$\dot{a} = 0 \qquad (2\text{-}9)$$

The vector $\bar{F}$ was formed as:

$$\bar{F} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \\ \dot{a} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ -\frac{\mu}{r^3}\, x + a \frac{v_x}{|\bar{v}|} \\ -\frac{\mu}{r^3}\, y + a \frac{v_y}{|\bar{v}|} \\ -\frac{\mu}{r^3}\, z = a \frac{v_z}{|\bar{v}|} \\ 0 \end{bmatrix} \qquad (2\text{-}10)$$

With an assumed set of initial conditions $(\bar{x}(t_0), t_0)$, the vector equation

$$\dot{\bar{X}} = \bar{F}\,(\bar{x}, t) \qquad (2\text{-}11)$$

can be numerically integrated to obtain a nominal trajectory. This trajectory is assumed to be known based on assumed initial conditions, but the true initial conditions differ slightly from those assumed, therefore

$$\bar{x}(t) = \bar{x}_0(t) + \delta\bar{x}(t) \tag{2-12}$$

where $\bar{x}(t)$ = the assumed nominal trajectory

$\bar{x}_0(t)$ = the true trajectory

$\delta\bar{x}(t)$ = the deviation from true trajectory

Differentiation gives

$$\frac{d}{dt}\bar{x}(t) = \frac{d}{dt}\bar{x}_0(t) + \frac{d}{dt}\delta\bar{x}(t) \tag{2-13}$$

which, together with eq 2-11 gives

$$\frac{d}{dt}\bar{x}_0(t) + \frac{d}{dt}\delta\bar{x}(t) = \bar{F}\,(\bar{x}_0(t) + \delta\bar{x}(t)) \tag{2-14}$$

where the time dependence of $\bar{F}$ has been incorporated into $\bar{x}(t)$. Consider the first component of eq 2-14:

$$\frac{d}{dt}x_o(t) + \frac{d}{dt}\delta x_o(t) = f_0\Big[x_0(t)+\delta\ x_0(t), x_1(t)+\delta x_1(t)\cdots x_n(t) + \delta x_n(t)\Big] \tag{2-15}$$

Letting t be fixed gives the right side of eq 2-15 as

$$f_0(x_0 + \alpha,\ x_1 + \beta \ldots . x_n + \gamma) \tag{2-16}$$

Application of Taylor's theorem gives

$$f_0(x_0+\alpha, x_1+\beta, \ldots x_n+\varepsilon+\gamma) = f_0\Big|_{\substack{x_0\\x_1\\\vdots\\x_n}} + \alpha\frac{\partial f_0}{\partial x_0}\Big|_{\substack{x_0\\x_1\\\vdots\\x_n}} + \beta\frac{\partial f_0}{\partial x_1}\Big|_{\substack{x_0\\x_1\\\vdots\\x_n}} \cdots + \gamma\frac{\partial f_0}{\partial x_n}\Big|_{\substack{x_0\\x_1\\\vdots\\x_n}}$$

$$+\ \mathcal{O}(2) \qquad (2\text{-}17)$$

where $\mathcal{O}(2)$ represents terms of second and higher orders which will be relatively small as long as α, $\beta, \ldots \gamma$ are small. So, to first order

$$\frac{d}{dt}\, x_0(t) + \frac{d}{dt}\delta \bar{x}_0(t) = f_0(x_0, x_1, \ldots x_n)$$

$$+ \frac{\partial f_0}{\partial x_0}\Big|_{\substack{x_0\\x_1\\\vdots\\x_n}} \cdot \delta x_0 + \frac{\partial f_0}{\partial x_1}\Big|_{\substack{x_0\\x_1\\\vdots\\x_n}} \cdot \delta x_1 + \ldots \frac{\partial f_0}{\partial x_n}\Big|_{\substack{x_0\\x_1\\\vdots\\x_n}} \cdot \delta x_n \qquad (2\text{-}18a)$$

In a similar manner, the second component of eq 2-15 is

$$\frac{d}{dt} x_1(t) + \frac{d}{dt}\delta x_1(t) = f_1(x_0, x_1 \ldots x_n)$$

$$+ \frac{\partial f_1}{\partial x_0}\Big|_{\substack{x_0\\x_1\\\vdots\\x_n}} \cdot \delta x_0 + \frac{\partial f_1}{\partial x_1}\Big|_{\substack{x_0\\x_1\\\vdots\\x_n}} \cdot \delta x_1 + \ldots + \frac{\partial f_1}{\partial x_n}\Big|_{\substack{x_0\\x_1\\\vdots\\x_n}} \cdot \delta x_n \qquad (2\text{-}18b)$$

But

$$\frac{d}{dt}\,\bar{x}(t) = \bar{F}(\bar{x}(t)) \qquad (2\text{-}11)$$

So recalling eq 2-14, eq 2-18 can be written as

$$\begin{bmatrix} \frac{d}{dt}\delta x_0(t) \\ \frac{d}{dt}\delta x_1(t) \\ \vdots \\ \frac{d}{dt}\delta x_n(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_0}{\partial x_0} & \frac{\partial f_0}{\partial x_1} & \cdots & \frac{\partial f_0}{\partial x_n} \\ \frac{\partial f_1}{\partial x_0} & \frac{\partial f}{\partial x_1} & \cdots & \vdots \\ \vdots & & & \vdots \\ \frac{\partial f_n}{\partial x_0} & \cdots\cdots & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\Bigg|_{\substack{x_0(t)\\ x_1(t)\\ \vdots \\ x_n(t)}} \begin{bmatrix} \delta x_0(t) \\ \delta x_1(t) \\ \vdots \\ \delta x_n(t) \end{bmatrix} \qquad (2\text{-}19)$$

So, to first order, $\delta\bar{x}(t)$ satisfies the time-varying linear differential equation:

$$\frac{d}{dt}\,\delta\bar{x}(t) = \underline{A}\left[\bar{x}_0(t)\right]\,\delta\bar{x}(t) \qquad (2\text{-}20)$$

where $\underline{A}\left[\bar{x}_0(t)\right]$ is defined by

$$\underline{A}_{i,j}\left[\bar{x}_0(t)\right] = \frac{\partial \bar{F}_i(\bar{x},t)}{\partial \bar{x}_j}\Bigg|_{\substack{t\,=\,\tau \\ x\,=\,\chi(\tau)}} \qquad (2\text{-}21)$$

$\underline{A}$, then, is a time-varying matrix whose elements are functions of $\bar{x}$ and t. The derivation of all elements of $\underline{A}$ is contained in Appendix A.

The elements of the matrix $\underline{A}$ were verified by conducting a numerical check given by:

$$A_{ij} \approx \frac{\bar{F}_i(\bar{x}_j+\delta,t) - \bar{F}_i(\bar{x},t)}{\delta} \qquad (2\text{-}22)$$

where:

A_{ij} = the element in the i^{th} row and j^{th} columns

δ = a deviation on the order of 10^{-5} or smaller

$\bar{F}_i(\bar{x},t)$ = the i^{th} component of $\bar{F}$, evaluated at $\bar{x}$

$\bar{F}_i(x_j+\delta,t)$ = the i^{th} component of $\bar{F}$, calculated when δ has been added to the j^{th} state of $\bar{x}$

In words, this check gives an approximation to the elements of $\underline{A}$ as a result of small changes in the state vector, $\bar{x}$. If the matrix, $\underline{A}$, derived numerically agrees with $\underline{A}$ evaluated at the state, $\bar{x}$, this provides assurance that the partial derivatives taken in deriving $\underline{A}$ are correct.

Eq 2-20 is linear, therefore solutions to it can be superimposed. Also, a fundamental set of n solutions (n vectors, $\bar{\phi}_i$) can be constructed from

$$\dot{\bar{\phi}}_i = \underline{A}\,\bar{\phi}_i(t_0) \qquad (2\text{-}23)$$

where $\bar{\phi}_i(t_0)$ is a vector having 1 as the i^{th} component and zeros elsewhere

The general solution to eq 2-20 is given by

$$\delta\bar{x}(t) = \delta x_1(t_0)\,\bar{\phi}_1(t) + \delta x_2(t_0)\bar{\phi}_2(t) + \ldots \delta x_n(t_0)\bar{\phi}_n(t) \qquad (2\text{-}24)$$

which can be assembled in matrix form as

$$\delta\bar{x}(t) - \underline{\Phi}(t,t_0)\,\delta\bar{x}(t_0) \qquad (2\text{-}25)$$

Since the columns, $\bar{\phi}_i$, of $\underline{\Phi}(t,t_0)$ satisfy eq 2-23, so does the matrix itself. The state transition matrix, $\underline{\Phi}$, is given by

$$\dot{\underline{\Phi}}(t,t_0) = \underline{A}(t\ )\underline{\Phi}(t,t_0) \tag{2-26}$$

and $\quad \underline{\Phi}(t,t_0) = \underline{I}$

where $\underline{I}$ = the identity matrix

III Derivation of Observation Relationships

The observation relationships for this problem were derived with both the observer and target vehicle in the same, x, y, z coordinate frame. The measurements from the observer were the two angles, azimuth and elevation as shown in Figure 1.

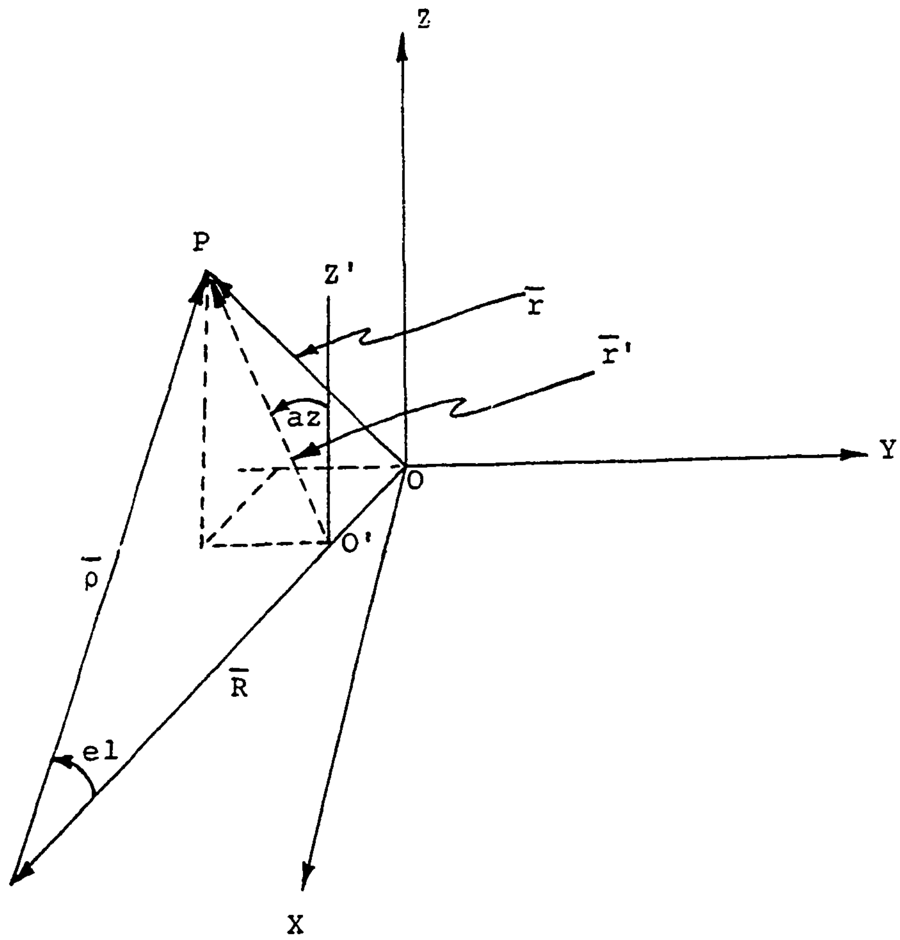
Azimuth was the angle measured in radians (positive in the clockwise direction) between a local vertical, z' and a local position vector, $\bar{r}'$.

Elevation was the angle, also in radians, between the negative of the position vector of the observer, $-\bar{R}$, and the vector from the observer to the target vehicle, $\bar{\rho}$.

The azimuth and elevation relationships were derived from the general equations:

$$\cos \gamma = \frac{a \cdot b}{|\bar{a}|\,|\bar{b}|} \tag{3-1}$$

$$\sin \gamma = \frac{|\bar{a} \times \bar{b}|}{|\bar{a}|\,|\bar{b}|} \tag{3-2}$$



$\bar{R}$ = position vector of observer

$\bar{r}$ = position vector of launch vehicle

$\bar{\rho}$ = position vector of launch vehicle relative to observer

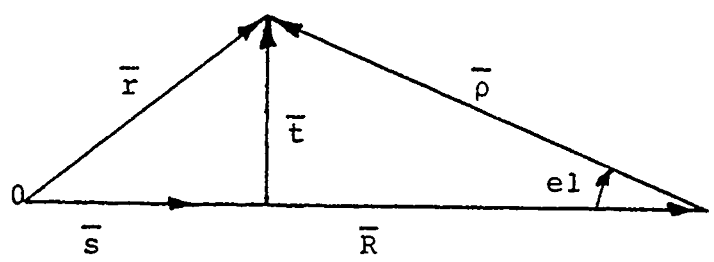
$\bar{r}'$ = position vector (in a plane orthogonal to $\bar{R}$) of launch vehicle relative to 0'

el = elevation angle; the angle subtended by $\bar{\rho}$ and $-\bar{R}$

az = azimuth angle; the angle subtended by r' and the line segment from 0' to z'

Figure 1. Illustration of observation angles

Elevation Derivation



$\overline{t}$ = a unit vector normal to $\overline{R}$ and containing point P

$\overline{s}$ = a unit vector along $\overline{R}$

Figure 2. Geometry for elevation derivation

Elevation is the angle between $\overline{\rho}$ and $-\overline{R}$, therefore from Figure 2 and eq 3-1

$$\cos\phi = \frac{|\overline{s}|}{|\overline{r}|} = \frac{\overline{s}\cdot\overline{r}}{|\overline{s}|\;|\overline{r}|} = \frac{\overline{R}\cdot\overline{r}}{|\overline{R}|\;|\overline{r}|} \tag{3-3}$$

From eq 3-3

$$|\overline{s}| = |\overline{r}|\cos\phi \tag{3-4}$$

Introducing direction to eq 3-4

$$\overline{s} = \frac{\overline{R}}{|\overline{R}|}\,|\overline{r}|\cos\phi \tag{3-5}$$

From vector addition

$$\overline{s} + \overline{t} = \overline{r} \tag{3-6}$$

which, combined with eq 3-5 gives

$$\frac{R}{|\overline{R}|}\,|\overline{r}|\cos\phi + \overline{t} = \overline{r} \tag{3-7}$$

Therefore

$$\bar{t} = \bar{r} - \frac{\bar{R}}{|\bar{R}|} |\bar{r}| \cos \phi \tag{3-8}$$

Combining results of eqs 3-3 and 3-8 gives

$$\bar{t} = \bar{r} - \frac{\bar{R}}{|\bar{R}|} |\bar{r}| \left[\frac{\bar{R} \cdot \bar{r}}{|\bar{R}|\,|\bar{r}|}\right] = \bar{r} - \bar{R}\left[\frac{\bar{R} \cdot \bar{r}}{|\bar{R}|\,|\bar{r}|}\right] \tag{3-9}$$

From Figure 2

$$\sin(el) = \frac{|\bar{t}|}{|\bar{\rho}|} \tag{3-10}$$

and

$$\bar{\rho} = \bar{r} - \bar{R} \tag{3-11}$$

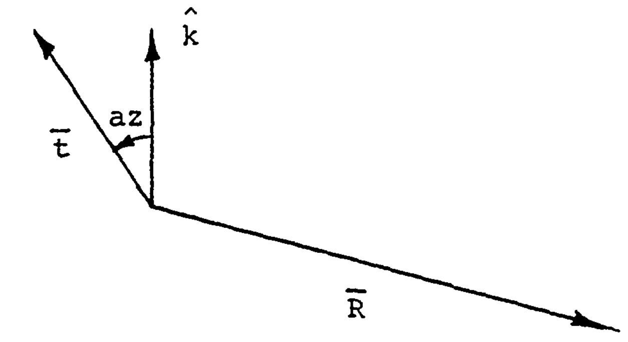
Therefore from eq 3-9

$$\sin(el) = \frac{\left[\bar{r} - \bar{R}\left(\frac{\bar{R} \cdot \bar{r}}{|\bar{R}|^2}\right)\right]}{|\bar{r} - \bar{R}|} \tag{3-12}$$

which gives

$$el = \sin^{-1}\left[\frac{\bar{r} - \bar{R}\left(\frac{\bar{R} \cdot \bar{r}}{|\bar{R}|^2}\right)}{|\bar{r} - \bar{R}|}\right] \tag{3-13}$$

Azimuth Derivation



$\bar{t}$ = vector the same as in Figure 2; normal to $\bar{R}$ and containing point P.

$\hat{k}$ = a unit vector in the z direction.

Figure 3. Geometry for azimuth derivation

Using eq 3-1 and Figure 3 gives

$$\cos(az) = \frac{\bar{t} \cdot \hat{k}}{|\bar{t}|\ |\hat{k}|} = \frac{\bar{t} \cdot \hat{k}}{|\bar{t}|} \tag{3-14}$$

Using eq 3-9 to expand for $\bar{t}$

$$\cos(az) = \frac{\left[\bar{r} - \bar{R}\left(\frac{\bar{R} \cdot \bar{r}}{|\bar{R}^2|}\right)\right] \cdot \hat{k}}{\left| r - \bar{R}\,\frac{\bar{R} \cdot \bar{r}}{|\bar{R}|^2} \right|} \tag{3-15}$$

So the equation for the azimuth is

$$az = \cos^{-1}\left[\frac{\left[\bar{r} - \bar{R}\left(\frac{\bar{R}\cdot\bar{r}}{|\bar{R}|^2}\right)\right]\cdot\hat{k}}{\left|\bar{r} - \bar{R}\left(\frac{\bar{R}\cdot\bar{r}}{|\bar{R}|^2}\right)\right|}\right] \tag{3-16}$$

The sign of the azimuth angle is ambiguous from eq 3-16, therefore when the equation was implemented in the filter, the sign was determined by looking at the z component of the cross product, $\bar{R} \times \bar{r}$. If the z component was negative, the azimuth was negative.

From the equations for the observation values, elevation and azimuth, it is seen that they are a nonlinear function of the first three elements of the state vector as well as the position vector of the observer. In reality, the observer's position would be known. Thus, the two element observation vector, $\bar{z}$ can be written as a function of the state vector:

$$\bar{z}(t_i) = \underline{G}\,(\bar{x}(t_i), t_i) \tag{3-17}$$

If $\bar{x}_0$ is the true state vector, a perfect observer would produce exact measurements:

$$\bar{z}_0(t_i) = \underline{G}(\bar{x}_0(t_i), t_i) \tag{3-18}$$

writing

$$\bar{x}_0(t_i) = \bar{x}(t_i) + \delta\bar{x}(t_i) \tag{3-19}$$

the true error, $\bar{e}_z(t_i)$, can be written:

$$\bar{e}_z(t_i) = \bar{z}(t_i) - \bar{z}_0(t_i) = \underline{G}(\bar{x}(t_i)t_i) - \underline{G}(\bar{x}_0(t_i), t_i) \quad (3\text{-}20a)$$

$$= \underline{G}(\bar{x}_0(t_i) + \delta\bar{x}(t_i), t_i) - \underline{G}(\bar{x}_0(t_i), t_i) \quad (3\text{-}20b)$$

$$= \left.\frac{\partial \underline{G}}{\partial \bar{x}}\right|_{x_0} \delta\bar{x}(t_i) \quad (3\text{-}20c)$$

Since we only have an estimate instead of the true state vector

$$\hat{x}(t_i) = \bar{x}_r(t_i) + \delta\bar{x}(t_i) \quad (3\text{-}21)$$

where $\hat{x}_r(t_i)$ is an estimate of the state vector and $\bar{x}_r(t_i)$ is a reference trajectory.

The residual is given by:

$$\bar{r}(t_i) = \bar{z}(t_i) - \bar{z}_r(t_i) = \underline{G}(\bar{x}_r(t_i) + \delta\bar{x}(t_i), t_i) - \underline{G}(\bar{x}_r(t_i), t_i) \quad (3\text{-}22)$$

which is approximated by

$$\bar{r}(t_i) \simeq \left.\frac{\partial \underline{G}}{\partial \bar{x}}\right|_{\bar{x}_r(t_i)} \delta\bar{x}(t_i) \quad (3\text{-}23)$$

Therefore, the residual, $\bar{r}(t_i)$, is linearly related to the correction, $\delta\bar{x}(t_i)$, between the reference state, $\bar{x}_r(t_i)$, and the state estimate, $\hat{x}(t_i)$. The linearizations are embodied in the $\underline{H}$ matrix given by

$$\underline{H}(t_i) = \frac{\partial \underline{G}(t_i)}{\partial \overline{x}(t_i)}\bigg|_{\overline{x}_r(t_i)} \tag{3-24}$$

Since $\underline{G}(t_i)$ is only a function of the three position elements of the state vector, the last four elements of each row of $\underline{H}$ are zero. The derivation of the first three elements of both rows of $\underline{H}$ is given in Appendix B.

In order to make the residual a function of the correction at t_0 instead of t_i, the state transition matrix is again introduced as in eq 2-25 .

Therefore

$$\overline{r}(t_i) \simeq \underline{H}(t_i)\delta\overline{x}(t_i) = \underline{H}(t_i)\underline{\Phi}(t_i,t_0)\delta\overline{x}(t_0) \tag{3-25}$$

The matrix product, $\underline{H}(t_i)\underline{\Phi}(t_i,t_0)$, is designated $\underline{T}\ (t_i)$.

The derivatives in $\underline{H}$ were verified in the same manner as those in $\underline{A}$. The verification resulted from looking at the equation

$$\underline{H}_i = \frac{\underline{G}(x_i+\Delta) - \underline{G}(\overline{x},t)}{\Delta} \tag{3-26}$$

where $\underline{H}_i$ = the i^{th} column of the H matrix

$\underline{G}(x_i+\Delta,t)$ = the evaluation of $\underline{G}$ with Δ added to the i^{th} state

$\underline{G}(\overline{x},t)$ = the evaluation of $\underline{G}$ with no Δ added to any of the states

Δ = a small value of the order 10^{-5}.

IV Seven - State Bayes Filter

This chapter presents the derivation of equations for the Bayes filter and the results of their implementation for the seven - state filter.

Derivation of Equations for Bayes Filter

The problem dynamics are given by the vector equation

$$\dot{\bar{x}} = \bar{F}(x,t) \qquad (2\text{-}11)$$

as derived in Chapter II. Because the dynamics of the problem are well understood, small deviations of the state vector at any time can be expressed as

$$\delta\bar{x}(t) = \underline{\Phi}(t,t_0)\ \delta\bar{x}(t_0) \qquad (2\text{-}25)$$

This expression is valid as long as the $\delta\bar{x}$'s are small.

Eq 2-25 was used to get an expression for the residual, $\bar{r}(t_i)$, as a linear function of the correction at the epoch time, $\delta\bar{x}(t_0)$

$$\bar{r}(t_i) \simeq \underline{H}(t_i)\underline{\Phi}(t_i,t_0)\delta\bar{x}(t_0)$$

$$\simeq \underline{T}(t_i)\delta\bar{x}(t_0) \qquad (3\text{-}25)$$

There is an error, due to imperfect measurements, between the true residual available if the true state vector were known, and the residual approximation in eq 3-25, therefore

$$\bar{r}(t_i) = \underline{T}(t_i)\delta\bar{x}(t_0)+\bar{e}(t_i) \qquad (4\text{-}1)$$

Rearranging, the error is expressed as

$$\bar{e}(t_i) = \bar{r}(t_i) - \underline{T}(t_i)\,\delta\bar{x}(t_0) \qquad (4\text{-}2)$$

Associated with each observation vector, $\bar{z}_i$, is a covariance matrix, $\underline{Q}_i$, containing information abcut the accuracy of the data instruments. If the measurements being processed are mutually independent, $\underline{Q}_i$ is a diagonal matrix. Using Gaussian error statistics, the probability of getting a particular error vector can be expressed as

$$f(\bar{e}) = (2\pi)^{-N/2}\ |\underline{Q}|^{-\frac{1}{2}}\ \exp\ (-\tfrac{1}{2}J) \qquad (4\text{-}3)$$

where N = the number of measurements

$\underline{Q}$ = the covariance matrix associated with the data

$J = \bar{e}^T\underline{Q}^{-1}\bar{e}$ and is a scalar

The principle of maximum likelihood is invoked by maximizing f. J is a quadratic form and must be minimized in order that f be a maximum. This is done by solving

$$\frac{\partial J}{\partial x} = \frac{\partial}{\partial x}\,(\bar{e}^T\underline{Q}^{-1}\bar{e}) = 0 \qquad (4\text{-}4)$$

for $\delta\bar{x}(t_0)$. Using the identity $(ab)^T = b^Ta^T$ and dropping the time dependence for clarity,

$$J = (\bar{r} - \underline{T}\delta\bar{x})^T\underline{Q}^{-1}(\bar{r} - \underline{T}\delta\bar{x})$$

$$= \bar{r}^T\underline{Q}^{-1}\bar{r} - \bar{r}^T\underline{Q}^{-1}\underline{T}\delta\bar{x} - \delta\bar{x}^T\underline{T}^T\underline{Q}^{-1}\bar{r} + \delta\bar{x}^T\underline{T}^T\underline{Q}^{-1}\underline{T}\delta\bar{x} \qquad (4\text{-}5)$$

Noting that $\frac{\partial}{\partial\bar{x}}(\bar{a}^T\bar{x}) = \frac{\partial}{\partial\bar{x}}(\bar{x}^T\bar{a}) = \bar{a}$, the partial derivative, $\frac{\partial J}{\partial\bar{x}}$, can be taken and set equal to zero:

$$\frac{\partial J}{\partial x} = 0-(\bar{r}^T\underline{Q}^{-1}\underline{T})^T - \underline{T}^T\underline{Q}^{-1}\bar{r} + (\delta\bar{x}^T\underline{T}^T\underline{Q}^{-1}\underline{T})^T + \underline{T}^T\underline{Q}^{-1}\underline{T}\delta\bar{x}$$

$$= -\underline{T}^T\underline{Q}^{-1}\bar{r} - \underline{T}^T\underline{Q}^{-1}\bar{r} + \underline{T}^T\underline{Q}^{-1}\underline{T}\delta\bar{x} + \underline{T}^T\underline{Q}^{-1}\underline{T}\delta\bar{x} \qquad (4-6)$$

(Note: $\underline{Q}$ is symmetric, therefore $(\underline{Q}^{-1})^T = \underline{Q}^{-1}$)

Setting $\frac{\partial J}{\partial \bar{x}}$ equal to zero and dividing by two yields

$$0 = -\underline{T}^T\underline{Q}^{-1}\bar{r} + \underline{T}^T\underline{Q}^{-1}\underline{T}\delta\bar{x} \qquad (4-7)$$

which gives

$$\delta\bar{x} = (\underline{T}^T\underline{Q}^{-1}\underline{T})^{-1}\underline{T}^T\underline{Q}^{-1}\bar{r} \qquad (4-8)$$

where the inverse, $(\underline{T}^T\underline{Q}^{-1}\underline{T})^{-1}$, exists as long as the states are observable over the interval of interest.

To fine the covariance associated with $\delta\bar{x}$, eq 4-8 can be rewritten as

$$\delta\bar{x} = \underline{W}\,\bar{r} + \bar{e}_x \qquad (4-9)$$

where $\underline{W} = (\underline{T}^T\underline{Q}^{-1}\underline{T})^{-1}\ \underline{T}^T\underline{Q}^{-1}$

Using this notation, the covariance of $\hat{\bar{x}}\ (t_0)$,

$$P_{\hat{x}}(t_0) = E\left[(\hat{\bar{x}} - \bar{x}_0)\ (\hat{\bar{x}} - \bar{x}_0)^T\right] \qquad (4-10)$$

is rewritten as

$$P_{\hat{x}}(t_0) = E\left[\underline{W}\left[(\bar{z} - \bar{z}_0)(\bar{z} - \bar{z}_0)^T\right]\underline{W}^T\right] \qquad (4-11)$$

and since $\underline{W}$ is deterministic,

$$P_{\hat{x}}(t_0) = \underline{W}\ E\left[(\bar{z} - \bar{z}_0)(\bar{z} - \bar{z}_0)^T\right]\underline{W}^T \qquad (4-12)$$

Since $\underline{Q}$ is the covariance of the measurements eq 4-12 can be written as

$$\underline{P}_{\hat{x}}(t_0) = \underline{WQW}^T \qquad (4-13)$$

Expanding and simplifying:

$$\underline{P}_{\hat{x}}(t_0) = (\underline{T}^T\underline{Q}^{-1}\underline{T})^{-1}\underline{T}^T\underline{Q}^{-1}\underline{Q}\left[(\underline{T}^T\underline{Q}^{-1}\underline{T})^{-1}\underline{T}^T\underline{Q}^{-1}\right]^T$$

$$= (\underline{T}^T\underline{Q}^{-1}\underline{T})^{-1}\underline{T}^T(\underline{T}^T\underline{Q}^{-1})^T\left[(\underline{T}^T\underline{Q}^{-1}\underline{T})^{-1}\right]^T$$

$$\underline{P}_{\hat{x}}(t_0) = (\underline{T}^T\underline{Q}^{-1}\underline{T})^{-1} \tag{4-14}$$

These equations can be altered somewhat to handle data in a sequential manner. If new data, $\bar{z}$, is to be added to an old estimate, $\hat{\bar{x}}(t_0^-)$, and its covariance, $\underline{P}(t_0^-)$, the old estimate can itself be treated as data. The observation relations for the estimate are

$$\bar{x} = \underline{I}\,\hat{\bar{x}}\,(t_0^-) \tag{4-15}$$

$$\bar{z} = \underline{G}\,(\bar{x}, t) \tag{4-16}$$

where $\underline{I}$ = the identity matrix

The augmented matrices are formed as

$$\underline{T}_{aug} = \begin{bmatrix} \underline{I} \\ \underline{T}_z \end{bmatrix} \tag{4-17}$$

$$\underline{Q}_{aug} = \begin{bmatrix} \underline{P}(t_0^-) & 0 \\ 0 & \underline{Q}_z \end{bmatrix} \tag{4-17b}$$

$$\bar{r}_{aug} = \begin{bmatrix} \left(\hat{\bar{x}}^- - \bar{x}_{ref}\right) \\ \bar{r}_z \end{bmatrix} \tag{417c}$$

where $\bar{x}_{ref}$ = the assumed reference state.

The 'z' subscript denotes a vector or matrix associated with the new data.

The 'aug' subscript denotes an augmented vector or matrix.

The updated covariance is given by

$$\underline{P}^{+} = (\underline{T}^{T}_{aug}\ \underline{Q}^{-1}_{aug}\ \underline{T}_{aug})^{-1}$$

$$= (\underline{P}^{-1}(-) + \underline{T}^{T}_{z}\ \underline{Q}^{-1}_{z}\ \underline{T}_{z})^{-1} \qquad (4\text{-}18)$$

The correction to the state vector is found similarly:

$$\delta\bar{x} = (\underline{T}^{T}_{aug}\ \underline{Q}^{-1}_{aug}\ \underline{T}_{aug})^{-1}\ \underline{T}^{T}_{aug}\ \underline{Q}^{-1}_{aug}\ \bar{r}_{aug}$$

$$= (\underline{P}^{-1}(-) + \underline{T}^{T}_{z}\underline{Q}^{-1}_{z}\underline{T}_{z})^{-1}\ (\underline{P}^{-1}(-)(\hat{\bar{x}}^{-} - \bar{x}_{ref}) + \underline{T}^{T}_{z}\underline{Q}^{-1}\bar{r}_{z}) \qquad (4\text{-}19)$$

Use of Equations in Filter Implementation

Starting with a good representation for the state vector as a reference solution, $\bar{x}_r(t_{i-1})$, integrate

$$\dot{\bar{x}} = \bar{F}(\bar{x}_r(t_{i-1}), t) \qquad (2\text{-}11)$$

to the time of the first measurement, t_i. Also integrate

$$\dot{\underline{\Phi}} = \underline{A}\ \underline{\Phi}(t, t_{i-1}) \qquad (2\text{-}26)$$

to get the state transition matrix from time t_{i-1} to t_i.

The residual, $\bar{r}(t_i)$, can be obtained from

$$\bar{r}(t_i) = \bar{z}(t_i) - \underline{G}(\bar{x}_r(t_i), t_i) \qquad (3\text{-}22)$$

Obtain current values for $\underline{T}(t_i)$ and $\underline{Q}$:

$$\underline{T}(t_i) = \underline{H}(t_i)\ \underline{\Phi}(t_i, t_{i-1}) \tag{3-25}$$

$$\underline{Q} = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} \quad \text{(constant for this implementation)}$$

The covariance is updated by including the information from the measurement:

$$\underline{P}(t_{i-1}^{+}) = \left[\underline{P}^{-1}(t_{i-1}^{-}) + \underline{T}^{T}(t_i)\underline{Q}_i^{-1}\underline{T}(t_i)\right]^{-1} \tag{4-18}$$

The correction to the reference state vector is given by

$$\delta\bar{x}(t_{i-1}) = \underline{P}(t_{i-1}^{+})\Big[\underline{P}^{-1}(t_{i-1}^{-})(\hat{x}^{-}(t_{i-1}) - \bar{x}_{ref}(t_{i-1})$$

$$+ \underline{T}(t_i)\underline{Q}_i^{-1}\bar{r}\ (t_i)\Big] \tag{4-19}$$

This correction is added to the reference state, $\bar{x}_r(t_{i-1})$, to get a new reference state. This procedure is continued until the elements of the state correction, $\delta\bar{x}(t_{i-1})$, and/or the residual, $\bar{r}(t_i)$, meet convergence criteria. When this occurs, the final $\bar{x}_r(t_{i-1})$ is integrated to time t_i, where it becomes the reference state to be integrated to the next measurement time. The covariance is also propagated forward in time.

$$\underline{P}\ (t_i^{-}) = \underline{\Phi}(t_i, t_{i-1})\ \underline{P}(t_{i-1}^{+})\ \underline{\Phi}^{T}(t_i, t_{i-1}) \tag{4-20}$$

The iterations again proceed, using the next observation.

The Bayes filter algorithm lends itself well to the performance parameter prediction problem. It can be started with only a guess for the reference state. The inverse of the covariance matrix, $\underline{P}^{-1}(t_i^-)$, can be initialized as a zero matrix, indicating there is no a priori knowledge of the system. This is actually the case when looking at launch data from a noncooperative ballistic missile. With $\underline{P}^{-1}(t_i^-)$ equal to zero the filter relies only on measurement data to make its corrections for the first set of measurements.

Although the Bayes algorithm is sequential, it can easily handle batches of data in a least squares mode. This flexibility was found to be useful for handling launch data and will be discussed further.

Initial Implementation of the Filter

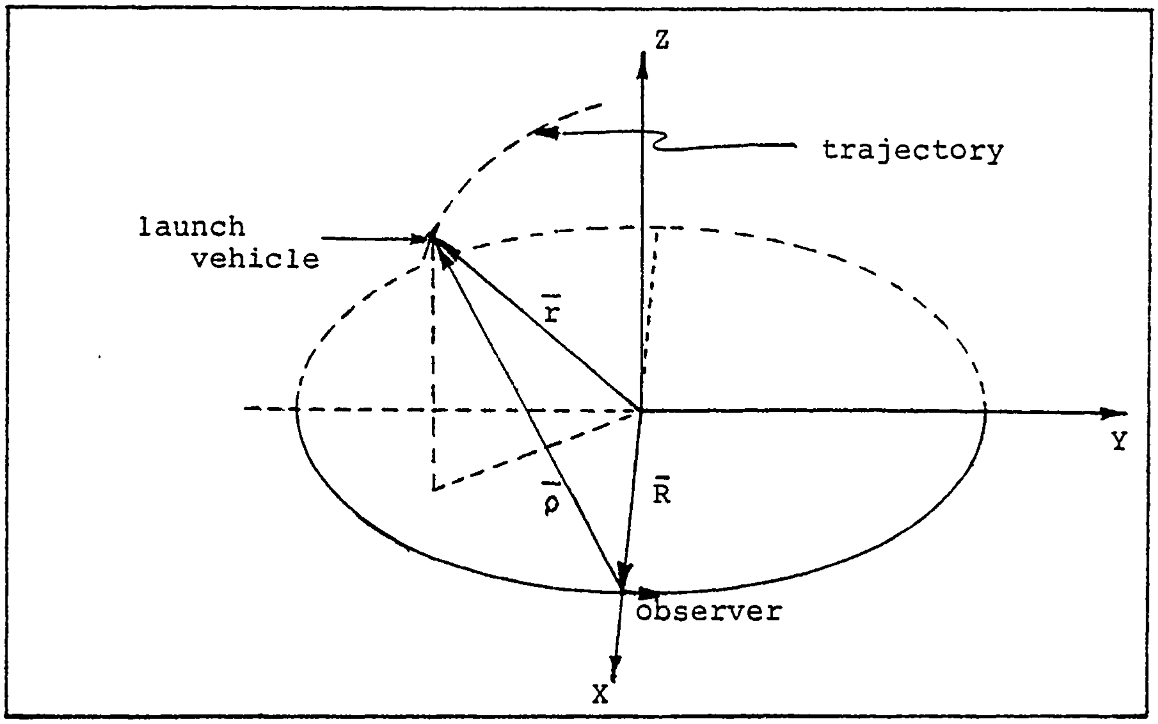


Figure 4. Initial positions of target and observer

The observer in this problem was assumed to be in a geosynchronous equatorial orbit. Tracking sequences were initialized with the observer positioned on the x axis as shown in Figure 4. This caused no loss in generality since the coordinate system chosen was arbitrary and the target was initialized from a point not also on any of the coordinate axes. During tracking, the observer progressed counterclockwise along its orbit.

Filter performance was checked by first determining its ability to determine zero-acceleration and constant acceleration before looking at increasing acceleration from launch data. The filter was able to detect and correct a 10^{-7} perturbation to the x position state on an unaccelerated trajectory as shown in Table 1. The correction was based on 8 measurements at 0.2 time unit (approximately 160 second) intervals.

With the same initial conditions the filter correctly detected a constant acceleration of 10^{-6} in addition to a perturbation to the x position state of 10^{-7}. This correction was based on 8 measurements at 0.2 time unit (approximately 160 second) intervals. Each measurement consisted of an elevation and an azimuth. It should be noted that for each of these cases the acceleration was constant which is how it was modeled in the filter. Also, the time interval between measurements was quite large. Time intervals of this magnitude are unacceptable for estimating acceleration during the ascent stages of a missile launch. The reason

Table I. Good correction to first state								
	Filter $\hat{x}$ initial	$\Delta\bar{x}$ from true	Corrections 1st	exp	2nd	exp	3rd	exp
1	.3026186	$+10^{-7}$	-.100004	-6	-.697	-11	-.603	-10
2	.8637757	0	.510483	-12	-.156	-10	.271	-10
3	.3026185	0	.152964	-12	-.475	-11	.218	-10
4	.7316202	0	-.732311	-11	-.282	-10	-.472	-10
5	.6471292	0	.168399	-11	.232	-10	.449	-11
6	.7316202	0	.120171	-11	.819	-11	.117	-10
7	.0	0	.234957	-11	.120	-10	.895	-11

Notes: Corrections based on 3 sets of 8 observations
Observations at 0.2 time unit (≃ 160 sec) intervals
Unaccelerated trajectory

Table II. Good correction to first and seventh states								
	Filter $\hat{x}$ initial	$\Delta\bar{x}$ from true	Corrections 1st	exp	2nd	exp	3rd	exp
1	.3026186	$+10^{-7}$	-.9999286	-7	-.398	-10	-.501	-10
2	.8637757	0	-.7829917	-12	-.926	-11	.249	-10
3	.3026185	0	-.2544198	-12	-.149	-11	.192	-10
4	.7316202	0	-.4013456	-12	-.550	-10	.362	-10
5	.6471292	0	-.1409622	-11	.297	-10	.156	-11
6	.7316202	0	-.1307843	-11	.135	-10	.845	-11
7	.0	-10^{-6}	.9999999	-6	.210	-10	.782	-11

Notes: Corrections based on 3 sets of 8 observations
Observations at 0.2 time unit (≃ 160 second) intervals
Trajectory acceleration constant -10^{-6}

for this is that over large time intervals on a launch, the vehicle acceleration would vary too much. This variance would make the filter's constant acceleration approximation less valid.

It was found that with just one observer the geometry of the problem takes on increased significance. A problem occurs when there is too little change in the elevation and azimuth measurements. This can occur if the time interval is too small, if the number of measurements is too few, or if the target vehicle is traveling away from the observer.

The geometry problem was highlighted during a zero-acceleration trajectory. The interval between measurements was 10 seconds and measurements were processed in batches of 35. The filter was unable to converge on a solution having zero acceleration. Instead, on each subsequent correction the magnitudes of the corrections were approximately twice the previous correction for each state as seen in Table III.

This suggests that the filter was unable to distinguish the target's position along the line of sight vector as shown in Figure 5. The true target position, $\bar{r}$, was indistinguishable from alternate target positions, $\bar{r}_i'$, which would result in the same values for elevation and azimuth.

To deal with the range observability problem a second observer was added to the filter to give it essentially

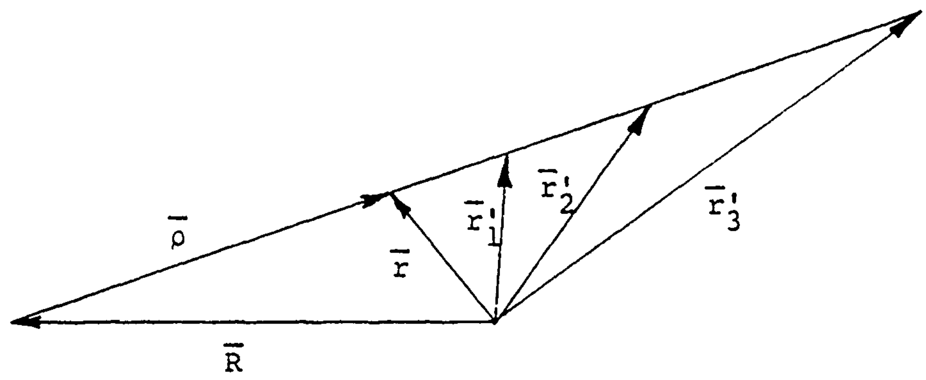
	Filter $\hat{x}_{initial}$	$\Delta\bar{x}$ from true	Corrections 1st	exp	2nd	exp	3rd	exp	4th	exp
Table III. Observability problem with one observer										
1	.5168881	0	-.572	-5	.430	-4	.442	-4	.817	-4
2	-.5144469	0	-.114	-5	.546	-5	.662	-5	.117	-4
3	.8695028	0	.231	-5	-.577	-5	-.557	-5	-.102	-4
4	.4330464	0	-.137	-3	-.138	-3	-.276	-4	-.552	-3
5	-.4125317	0	-.168	-4	-.822	-5	-.228	-4	-.467	-4
6	.7156707	0	.225	-4	.142	-4	.346	-4	.701	-4
7	0.0	+.001	-.947	-3	.409	-4	.919	-4	.185	-3

Notes: Corrections based on 35 observations
Observations at 10 second intervals
Zero acceleration trajectory

Table IV. Increased observability with two observers.

	Filter $\hat{x}_{initial}$	$\Delta\bar{x}$ from true	1st correction	exp
1	.5168881	0	-.1695	-6
2	-.5144469	0	-.3243	-6
3	.8695028	0	.5529	-7
4	.4330464	0	.2563	-5
5	-.4123517	0	.4691	-5
6	.7156707	0	-.6703	-6
7	0.0	+.001	-.9954	-3

Notes: Corrections based on 35 observations
Observations at 10 second intervals
Zero acceleration trajectory



$\bar{R}$ = observer position vector

$\bar{r}$ = true target vehicle position vector

$\bar{r}'_{1-3}$ = alternate target vehicle position vectors

$\bar{\rho}$ = vector from observer to target vehicle

Figure 5. Measurement Ambiguity with One Observer

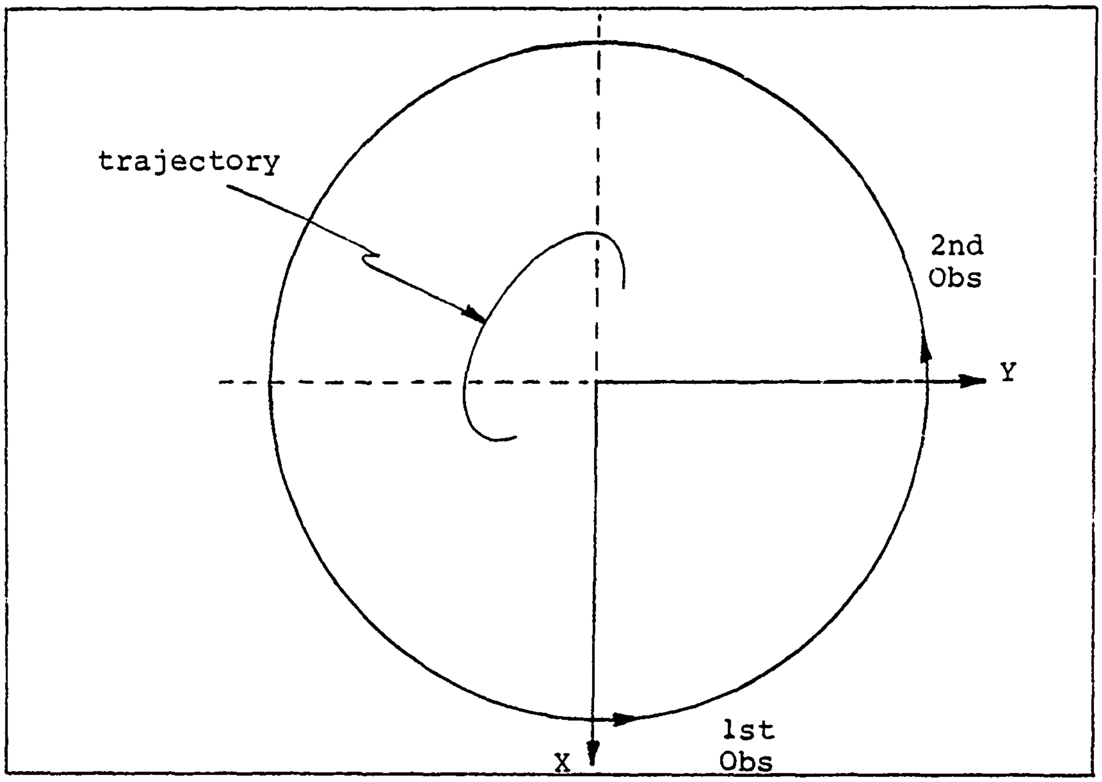


Figure 6. Placement of 1^{st} and 2^{nd} Observers

a "stereo" view of the target vehicle's location. The second observer was assumed to be in a geosynchronous equitorial orbit, but was positioned 90 degrees ahead of the first observer as shown in Figure 6.

With the same initial conditions the trajectory was again run with the addition of the second observer. The filter essentially corrected to the correct solution in one iteration as shown by Table IV.

One more trajectory with constant acceleration was tested before proceeding to launch trajectories. This final constant acceleration trajectory started from initial conditions similar to those for a launch trajectory. Data was input every second and ten measurements were processed simultaneously. The filter was allowed to iterate using the first batch of data to see if it would converge on a constant acceleration solution. The first correction was best, after which subsequent corrections shifted away from the proper solution as seen in Table V. Of particular interest was the instability demonstrated in the 10^{th} and 11^{th} corrections. These two corrections were normalized and their dot product was -0.999991289 which indicates the vectors were parallel. Increasingly greater corrections with opposite signs were being made along the same vector in state space, characteristic of an oscillating divergence. The matrix, $\underline{T}^T \underline{Q}^{-1} \underline{T}$, became noninvertable following the 11^{th} correction. The reason for this instability was not determined.

Table V. Demonstrated Filter Instability (7-state)

	Filter $\hat{x}_{initial}$	exp	$\Delta\bar{x}$ from true	Corrections 1st	exp	2nd	exp	3rd	exp	4th	exp	5th	exp
1	.4545195	0	0	.3907	-8	.4721	-8	.8740	-8	.1778	-7	.3657	-7
2	-.4545194	0	0	-.3894	-8	-.4668	-8	-.8594	-8	-.1740	-7	-.3567	-7
3	.7660445	0	0	-.7906	-8	-.6559	-8	-.1432	-7	-.2817	-7	-.5464	-7
4	.4545195	-3	0	.1812	-5	.6136	-6	.2372	-5	.4572	-5	.8504	-5
5	-.4545194	-3	0	-.1812	-5	-.6140	-6	-.2374	-5	-.4574	-5	-.8511	-5
6	.7660945	-3	0	.3160	-5	.1140	-5	.4213	-5	.8141	-5	.1523	-4
7	1.2	0	-.2	.1988	0	-.3849	-3	-.1543	-2	-.2994	-2	-.5646	-2

Corrections

	6th	exp	7th	exp	8th	exp	9th	exp	10th	exp	11th	exp
1	.7684	-7	.1677	-6	.4013	-6	.1198	-5	.4484	-5	-.1401	-5
2	.7473	-7	-.1629	-6	-.3903	-6	-.1174	-5	-.4447	-5	.1493	-5
3	-.1030	-6	-.1825	-6	-.2557	-6	.1829	-6	.5077	-5	-.8108	-5
4	.1483	-4	.2051	-4	.6152	-5	-.1593	-3	-.1533	-2	.1895	-2
5	-.1465	-4	-.2055	-4	-.6242	-5	.1592	-3	.1535	-2	-.1891	-2
6	.2653	-4	.3850	-4	.1813	-4	-.2635	-3	-.2769	-2	.2923	-2
7	-.1002	-1	-.1544	-1	-.1405	-1	.4365	-1	.6616	0	-.3454	1

Notes: Corrections based on 10 observations
Observations at 1 second intervals
Trajectory acceleration constant - 1.4

Based on the filter's consistent ability to calculate a good first correction with constant acceleration data, filter performance with variable acceleration data was investigated. Data consisted of elevation and azimuth measurements as described in Appendix C generated for various time intervals. Because acceleration was modeled as constant in the filter, compensation had to be made to enable the seventh state to change with varying acceleration.

An ad hoc fading memory was added to the filter in which elements of the covariance matrix were deweighted to reflect decreased confidence in the seventh state. This was achieved by pre-and post-multiplying the inverse covariance matrix, $\underline{P}^{-1}(t_i^-)$, by a diagonal matrix just after propagation. The elements of the diagonal deweighting matrix consisted of 0.99 for the first three elements, 0.98 for the next three elements, and the seventh element taking on various values less than 1 to vary filter performance. (ref:8)

For filter attempts at estimating variable acceleration, measurements were provided at one second intervals and were introduced to the filter in sets of five measurements. Sets of five were used because of the ease of implementation. As a lower bound, the filter had to have an initial set of at least four measurements in order to make all seven states observable. As an upper bound, the larger the number of measurements per set, the more the acceleration would vary over the longer time

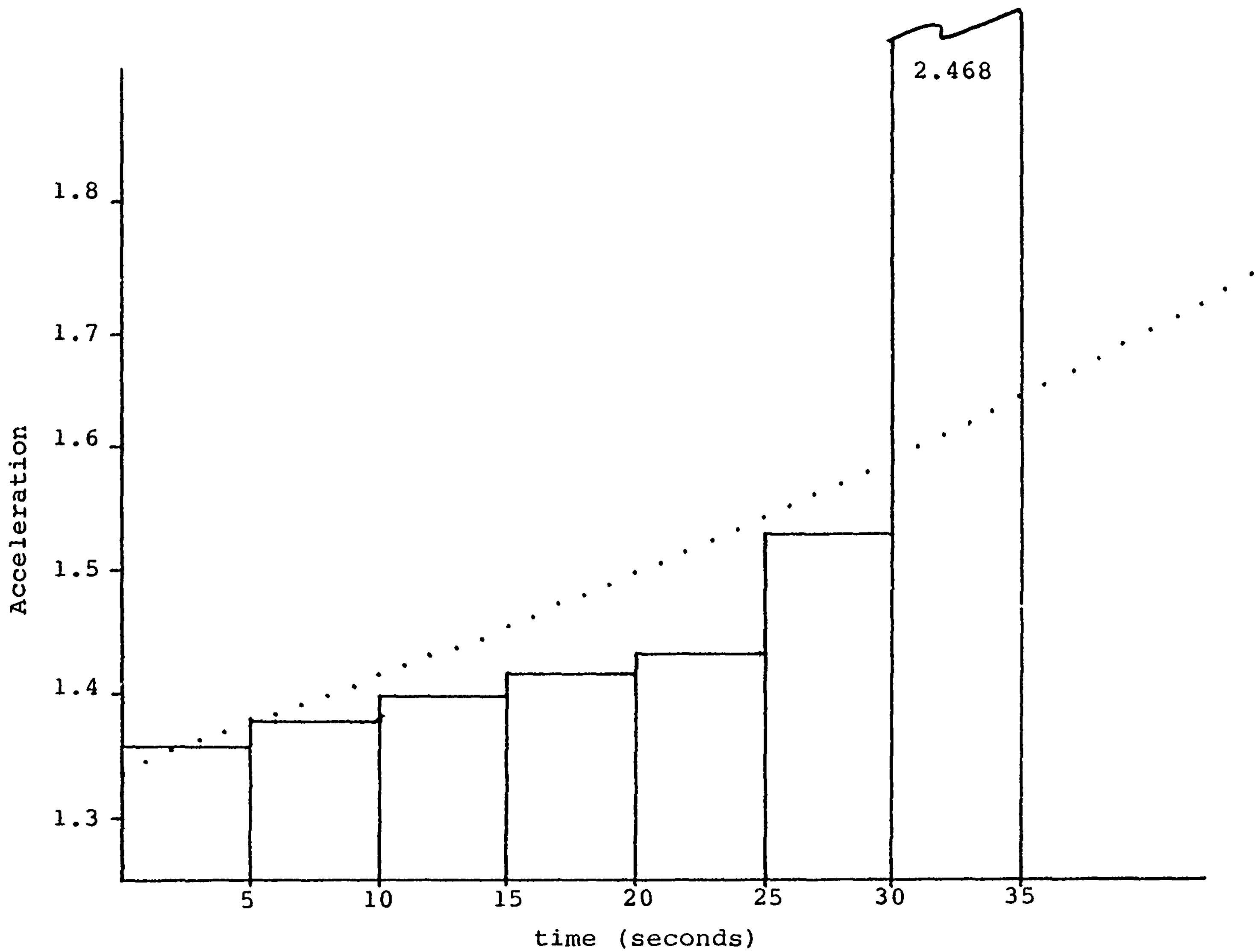


Figure 7. Estimated acceleration from fading memory filter

interval. This would make the filter's constant acceleration approximation less valid.

Regardless of the value selected for the seventh element of the deweighting matrix, the filter estimates were similar. The first five acceleration estimates increased linearly with values less than the actual values as shown in Figure 7. The filter would then respond to the increasing residuals, overcompensating with excessive estimates for acceleration.

In view of these results an attempt was made to implement a better model of acceleration in the filter. An additional state was required to do this. The development and results are contained in the following chapter.

V The Eight - State Filter Model

Because of the instability demonstrated by the seven - state filter, the filter representation for acceleration was changed to get a better depiction of reality. Treating acceleration as constant between updates was attractive because it would have been conceptually general enough to allow the filter to process data from the ascent of a launch vehicle as well as from the descent of a reentry vehicle. The performance of such a filter negated the possibility of using sc general a model.

Derivation of Equations

If thrust is assumed constant for each stage of a launch vehicle in accordance with usual theory (Ref 4:369), the acceleration of the vehicle can be described by the equation

$$a_T = V_e \frac{\dot{m}}{(m_0 - \dot{m}t)} \qquad (5\text{-}1)$$

where a_T = acceleration due to the thrust

$\dot{m}$ = the propellant mass flow rate (known as β in some literature)

m_0 = the original mass of the missile (including propellant)

t = time in any consistent units

If implemented in this manner, the filter would gain three new states - V_e, $\dot{m}$, and m_0. However, if the numerator and denominator are divided by m_0, a_T becomes:

$$a_T = V_e \frac{\frac{\dot{m}}{m_0}}{(1-\frac{\dot{m}}{m_0} t)} = V_e \frac{M}{(1-Mt)} \tag{5-2}$$

where M = the relative mass flow rate.

This reduces the number of additional states to two, making an eight-state filter with the state vector:

$$\bar{x} = \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \\ V_e \\ M \end{bmatrix} \tag{5-3}$$

The resulting equations of motion are:

$$\dot{\bar{x}} = \bar{F}\ (\bar{x}(t),t) = \begin{bmatrix} v_x \\ v_y \\ v_z \\ -x/R^3 + a_T\ v_x/v \\ -y/R^3 + a_T\ v_y/v \\ -z/R^3 + a_T\ v_z/v \\ 0 \\ 0 \end{bmatrix} \tag{5-4}$$

where, as before: $R = x^2 + y^2 + z^2$

$$v = v_x^2 + v_y^2 + v_z^2$$

$$a_T = V_e \frac{M}{1-Mt}$$

Further, the new states result in changes to the $\underline{A}$ matrix. Aside from $\underline{A}$ now being an 8x8 matrix, there are six new non-zero terms:

$$A_{4,7} = \frac{v_x a_T}{v V_e} \tag{5-5}$$

$$A_{4,8} = \frac{v_x}{v}\left(\frac{a_T^2 t}{V_e M} + \frac{a_T}{M}\right) \tag{5-6}$$

$$A_{5,7} = \frac{v_y a_T}{v V_e} \tag{5-7}$$

$$A_{5,8} = \frac{v_y}{v}\left(\frac{a_T^2 t}{V_e M} + \frac{a_T}{M}\right) \tag{5-8}$$

$$A_{6,7} = \frac{v_z a_T}{v V_e} \tag{5-9}$$

$$A_{6,8} = \frac{v_z}{v}\left(\frac{a_T^2 t}{V_e M} + \frac{a_T}{M}\right) \tag{5-10}$$

The added elements to both $\underline{G}$ and $\underline{H}$ due to the additional state are zeros. Other than modifying matrix dimensions and indices for compatibility with the eighth state, no further changes to the filter were required.

Filter Checkout and Performance

The filter was initially given data with one second between observations (at each observation time the filter was given elevation and azimuth data from both observers). When these data were processed to produce a least squares estimate, both exit velocity, V_e, and relative mass flow rate, M, were observable. However, when the time interval between observations was increased to ten seconds, the resulting summation of $\underline{T}^T \underline{Q}^{-1} \underline{T}$ was ill-conditioned and couldn't be inverted. The diagonal elements corresponding to the position states were of the order 10^{13}; those for velocity were of the order 10^{18}; while those for V_e and M were 10^2 and 10^{12}, respectively.

As a result of these numerical difficulties, the seventh and eighth states were reinspected for possible changes. Magnitudes of V_e were typically 200 to 300 while those for M were approximately 0.005. Magnitudes for position and velocity were from 0 to 2. By multiplying V_e by M in the numerator of the acceleration expression, the term is changed from

$$a_t = V_e \frac{M}{(1+Mt)}$$

to

$$a_T = \frac{V'}{1+Mt} \qquad (5\text{-}11)$$

where $V' = V_e \times M$

This new parameter has dimensions typical of an acceleration and magnitudes in the range of 1 - 2. When this parameter was adopted into the filter and the same data was run as before, the estimate was again stopped by an ill-conditioned covariance matrix. The diagonal elements of the last two states were both of the order 10^7, while those corresponding to position and velocity were as high as before.

The same filter configuration was then given data from the second stage of the ascent. Thirteen observations were processed simultaneously. The covariance matrix was invertible, but the filter correction was unsatisfactory. The filter was not able to identify and correct perturbations to the initial conditions of the states. The corrections indicated that the filter was attempting to reduce the residuals by correcting all states an equal amount, instead of just the ones which had been perturbed from nominal values.

Based on this result, observation data was generated which attempted to accentuate the quadratic effects on position due to acceleration in contrast to the linear effects due to velocity. To clarify this somewhat, if residuals were calculated for a trajectory in which the initial position states had been perturbed, there would essentially be a constant displacement in the calculated observations. This would result in residual values which were proportional to the original position perturbation:

$$r \propto \Delta p \tag{5-12}$$

where r = residual

Δp = perturbation in position

In contrast to this, if the initial velocity states were perturbed, the residuals could be expected to grow linearly with time:

$$r \propto \Delta v \, \Delta t \qquad (5\text{-}13)$$

where Δt = the change in time

Δv = perturbation in velocity

And finally, with perturbation of the initial acceleration state(s), the residuals would increase approximately quadratically with time:

$$r \propto \Delta a \, (\Delta t)^2 \qquad (5\text{-}14)$$

where Δa = perturbation in acceleration

Therefore, for small trajectory arcs (small Δt) the filter could resolve the residuals by changing position, velocity, acceleration, or any combination of the three. Alternately, for long arcs of data, acceleration effects should be dominant. With this reasoning, a trajectory was generated which varied in acceleration from a value of 1.09 to 20. With small corrections to either the seventh or eighth state the filter calculated corrections for all states as seen in Tables VI and VII. Given the correct values for the two acceleration states and with the fourth state perturbed the filter still made inappropriate corrections as shown in Table VIII. The reason for these problems was not determined.

Table VI. 8-state filter, 7th state perturbed						
	Filter $\hat{x}$ initial	$\Delta\bar{x}$ from true	Corrections 1st	exp	2nd	exp
1	1.1045353	0	-.39607951	-6	-.14425749	-6
2	-.6377037	0	-.10541036	-7	-.24694020	-6
3	0	0	-.53911275	-7	-.26563227	-9
4	.3391559	0	.27334082	-6	.16089711	-6
5	.5874354	0	.40171909	-6	.69744367	-6
6	.5691713	0	.48656250	-6	.63308899	-6
7	1.090001	$+10^{-6}$	-.15398878	-5	-.10089342	-5
8	.00051893	0	.10604722	-9	1.27149168	-9

Notes: 180 observations processed simultaneously
Same data for both corrections
10 second interval between observations
Acceleration increased from 1.09 to 20

Table VII. 8-state filter, 8th state perturbed						
	Filter $\hat{x}$ initial	$\Delta\bar{x}$ from true	Corrections 1st	exp	2nd	exp
1	1.1045353	0	.44489853	-3	.17403788	-3
2	-.6377037	0	.13866326	-4	.15267806	-3
3	0	0	-.27868686	-5	.29973877	-4
4	.3391559	0	-.46312445	-3	-.36587537	-3
5	.5874354	0	.52541563	-4	-.34485975	-3
6	.5691713	0	.26822451	-4	-.23568659	-3
7	1.09	0	.49368101	-4	.67655352	-3
8	.00051993	$+10^{-6}$	-.10231345	-5	-.71595424	-7

Notes: 180 observations processed simultaneously
Same data for both corrections
10 second interval between observations
Acceleration increased from 1.09 to 20

Table VIII. 8-state filter, 4th state perturbed						
	Filter $\hat{x}_{initial}$	$\Delta\bar{x}$ from true	Corrections			
			1st	exp	2nd	exp
1	1.1045353	0	-.12170983	-5	1.79428858	-6
2	-.6377037	0	.81175116	-7	-.75370992	-7
3	0	0	-.27062575	-7	-.90027939	-7
4	.3391569	$+10^{-6}$	.14454243	-5	.22434660	-5
5	.5874354	0	.34219555	-6	.86850929	-6
6	.5691713	0	.21885328	-7	.37698359	-6
7	1.09	0	-.21938892	-5	-.29797983	-5
8	.00051893	0	.39798295	-9	.64701346	-9

Notes: 180 observations processed simultaneously
Same data for both corrections
10 second interval between observations
Acceleration increased from 1.09 to 20

VI Conclusions and Recommendations

Conclusions

(1) The seven-state (3 states for position, 3 states for velocity, and 1 for acceleration) Bayes filter can estimate acceleration of a launch vehicle if the acceleration is constant (including zero acceleration).

(2) Study results indicate that variable acceleration cannot be estimated with the seven-state Bayes filter employing a fading memory to compensate for the constant acceleration modelled in the seventh state.

(3) Results indicate that variable acceleration cannot be estimated with an eight-state Bayes filter with acceleration modelled using engine exit velocity, launch vehicle mass, and propellant mass flow rate.

(4) If measurements from only one observer are available, data arcs yielding small changes in the measurement angles may result in an unobservable system. The short data arcs can be the result of too short a time interval of observation or problem geometry.

Recommendations

Some alternate methods need to be studied which specifically take advantage of the type of measurements available in this problem. The azimuth and elevation measurements should result in a good position time - history for a launch vehicle. Two possible approaches to using this information follow.

One possibility might be to alter the existing structure of the filter algorithms used in this study. Typically, at the initiation of a launch, the launch position is well known. The velocity starts at zero, but its direction is not well known a priori. The acceleration can be said to be greater than one but is otherwise unknown. The basic idea would be to correct the states according to their expected contribution to the residuals. Therefore, the acceleration state(s) would be corrected (ignoring the other six states) on the initial filter iterations(s). The velocity and position could be corrected subsequently (velocity corrected before position). If the launch location was assumed known, position wouldn't be corrected at all.

Another alternative would be to use a smoothing process. Elevation and azimuth data would be used to get a time-history launch vehicle position. A smoothing process (ref:5) could be used to get a smooth curve which would be differentiated analytically or numerically to get velocity and then acceleration. If numerical differentiation is used to get velocity, the velocity curve would also have to be smoothed before differentiating to get acceleration. In addition the acceleration might also need smoothing to get a good representation of acceleration for all times of interest.

Further analysis should be conducted to investigate the causes of some of the problems encountered during this study. With regard to observability problems, range measurements might be added to see if the problems disappear. If so, the deletion of range measurements and addition of another observer could be tried to see if similar results are obtained. If, instead, the addition of range measurements doesn't affect observability, further analysis would have to be conducted to look for the real cause of problems.

Special attention should be paid to the time intervals of observation. Long term propagation of the state transition matrix may be erroneous. This could then be the source of many problems.

With the seven state filter, some alternate fading memory techniques might be attempted. Also, the use of pseudo-noise was not tried in this effort because of the desire to keep the model simple. In view of results obtained thus far, pseudo-noise addition is a possibility which might be investigated.

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Appendix A

Derivation of the A Matrix

The elements of the A matrix are found by taking the gradient of the dynamics matrix (vector), $\overline{F}$.

$$\underline{A} = \overline{\nabla F} \tag{A-1}$$

where the elements are given by

$$A_{ij} = \frac{\partial \overline{F}_i}{\partial \overline{x}_j} \tag{A-2}$$

Therefore:

$$\underline{A} = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} & \frac{\partial \dot{x}_1}{\partial x_3} & \frac{\partial \dot{x}_1}{\partial x_4} & \frac{\partial \dot{x}_1}{\partial x_5} & \frac{\partial \dot{x}_1}{\partial x_6} & \frac{\partial \dot{x}_1}{\partial x_7} \\ \frac{\partial \dot{x}_2}{\partial x_1} & & & & & & \vdots \\ \frac{\partial \dot{x}_3}{\partial x_1} & & & & & & \vdots \\ \frac{\partial \dot{x}_4}{\partial x_1} & & & & & & \vdots \\ \frac{\partial \dot{x}_5}{\partial x_1} & & & & & & \vdots \\ \frac{\partial \dot{x}_6}{\partial x_1} & & & & & & \vdots \\ \frac{\partial \dot{x}_7}{\partial x_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial \dot{x}_7}{\partial x_7} \end{bmatrix} \tag{A-3}$$

Using the state equations given in Chapter II, the nonzero elements of A are:

$$A_{14} = \frac{\partial \dot{x}}{\partial v_x} = 1 \tag{A-4}$$

$$A_{25} = \frac{\partial \dot{y}}{\partial v_y} = 1 \tag{A-5}$$

$$A_{36} = \frac{\partial \dot{z}}{\partial u_z} = 1 \tag{A-6}$$

$$A_{41} = \frac{\partial \dot{v}_x}{\partial x} = \frac{-\mu}{r^3} + \frac{3\mu x^2}{r^5} \tag{A-7}$$

$$A_{42} = \frac{\partial \dot{v}_x}{\partial y} = \frac{3\mu xy}{r^5} \tag{A-8}$$

$$A_{43} = \frac{\partial \dot{v}_x}{\partial z} = \frac{3\mu xz}{r^5} \tag{A-9}$$

$$A_{44} = \frac{\partial \dot{v}_x}{\partial v_x} = \frac{-v_x^2 a}{v^3} + \frac{a}{v} \tag{A-10}$$

$$A_{45} = \frac{\partial \dot{v}_x}{\partial v_y} = \frac{-v_x v_y a}{v^3} \tag{A-11}$$

$$A_{46} = \frac{\partial \dot{v}_x}{\partial v_z} = \frac{-v_x v_z a}{v^3} \tag{A-12}$$

$$A_{47} = \frac{\partial \dot{v}_x}{\partial t} = \frac{v_x}{v} \tag{A-13}$$

$$A_{51} = \frac{\partial \dot{v}_y}{\partial x} = \frac{3\mu yx}{r^5} \tag{A-14}$$

$$A_{52} = \frac{\partial \dot{v}_y}{\partial y} = \frac{-\mu}{v^3} + \frac{3\mu y^2}{r^5} \tag{A-15}$$

$$A_{53} = \frac{\partial \dot{v}_y}{\partial z} = \frac{3\mu yz}{r^5} \tag{A-16}$$

$$A_{54} = \frac{\partial \dot{v}_y}{\partial v_x} = \frac{-v_y v_x a}{v^3} \tag{A-17}$$

$$A_{55} = \frac{\partial \dot{v}_y}{\partial v_y} = \frac{-v_y^2 a}{v^3} + \frac{a}{v} \tag{A-18}$$

$$A_{56} = \frac{\partial \dot{v}_y}{\partial v_z} = \frac{-v_y v_z a}{v^3} \tag{A-19}$$

$$A_{57} = \frac{\partial \dot{v}_y}{\partial t} = \frac{v_y}{v} \tag{A-20}$$

$$A_{61} = \frac{\partial \dot{v}_z}{\partial x} = \frac{3\mu z x}{r^5} \tag{A-21}$$

$$A_{62} = \frac{\partial \dot{v}_z}{\partial y} = \frac{3\mu z y}{r^5} \tag{A-22}$$

$$A_{63} = \frac{\partial \dot{v}_z}{\partial z} = -\frac{\mu}{r^3} + \frac{3\mu z^2}{r^5} \tag{A-23}$$

$$A_{64} = \frac{\partial \dot{v}_z}{\partial v_x} = \frac{-v_z v_x a}{v^3} \tag{A-24}$$

$$A_{65} = \frac{\partial \dot{v}_z}{x_y} = \frac{-y_z v_y a}{v^3} \tag{A-25}$$

$$A_{66} = \frac{\partial v_z}{\partial v_z} = \frac{-v_z^2 a}{v^3} + \frac{a}{v} \tag{A-26}$$

$$A_{67} = \frac{\partial \dot{v}_z}{\partial t} = \frac{v_z}{v} \tag{A-27}$$

where

$$r = \sqrt{x^2+y^2+z^2}$$

$$v = \sqrt{v_y^2+v_y^2+v_z^2}$$

and other variables are as defined in Chapter II

Appendix B

Derivation of H Matrix

The elements of the H matrix are found by taking the partial derivatives of elevation and azimuth with respect to all elements of the state vector. $\underline{H}$ is therefore a 2 by 7 matrix given by:

$$\underline{H} = \frac{\partial \underline{G}}{\partial \bar{\underline{x}}} = \begin{bmatrix} \frac{\partial el}{\partial x} & \frac{\partial el}{\partial y} & \frac{\partial el}{\partial z} & \frac{\partial el}{\partial v_x} & \frac{\partial el}{\partial v_y} & \frac{\partial el}{\partial v_z} & \frac{\partial el}{\partial a} \\ \frac{\partial az}{\partial x} & \frac{\partial az}{\partial y} & \frac{\partial az}{\partial z} & \frac{\partial az}{\partial v_x} & \frac{\partial az}{\partial v_y} & \frac{\partial az}{\partial v_z} & \frac{\partial az}{\partial a} \end{bmatrix} \quad \text{(B-1)}$$

The last four columns of the matrix are zero.

Partial derivatives of elevation:

$$el = \sin^{-1}\left[\frac{\left| -\bar{R}\dfrac{(\bar{r}\cdot\bar{R})}{|\bar{R}|\,|\bar{R}|} + \bar{r}\right|}{\left|\bar{r}-\bar{R}\right|}\right] \quad \text{(B-2)}$$

$$\frac{d}{dx}\left(\sin^{-1}u\right) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \quad \text{(B-3)}$$

$$|\bar{r}| = \sqrt{x^2+y^2+z^2} \quad \text{(B-4)}$$

where x,y,z are position components for the target vehicle

$$|\bar{R}| = \sqrt{R_x^2+R_y^2+R_z^2} \quad \text{(B-5)}$$

where R_x, R_y, R_z are position components of the observer.

Using eqs (B-2) and (B-3):

$$\frac{\partial el}{\partial x} = \left[1 - \left[\frac{\left| \frac{-\bar{R}(\bar{r} \cdot \bar{R})}{|\bar{R}|\,|\bar{R}|} + \bar{r} \right|}{|\bar{r} - \bar{R}|} \right]^2 \right]^{-\frac{1}{2}} \frac{d}{dx} \left[\frac{\left| \frac{-\bar{R}(\bar{r} \cdot \bar{R})}{|\bar{R}|\,|\bar{R}|} + \bar{r} \right|}{|\bar{r} - \bar{R}|} \right] \tag{B-6}$$

$$\frac{\partial el}{\partial x} = \left[1 - \left(\frac{\left| \frac{-\bar{R}(\bar{r} \cdot \bar{R})}{|\bar{R}|\,|\bar{R}|} + \bar{r} \right|}{|\bar{r} - \bar{R}|} \right)^2 \right]^{-\frac{1}{2}} \left[(-1) \left(\left| \frac{-\bar{R}(\bar{r} \cdot \bar{R})}{|\bar{R}|\,|\bar{R}|} + \bar{r} \right| \right) \left(|\bar{r} - \bar{R}| \right)^{-2} + \left| \frac{-\bar{R}(R_x)}{|\bar{R}|\,|\bar{R}|} + 1 \right| \left(|\bar{r} - \bar{R}| \right)^{-1} \right] \tag{B-7}$$

Similarly:

$$\frac{\partial el}{\partial y} = \frac{\left[\frac{(-1) \left(\left| \frac{-\bar{R}(\bar{r} \cdot \bar{R})}{|\bar{R}|\,|\bar{R}|} + \bar{r} \right| \right)}{\left(|\bar{r} - \bar{R}| \right)^2} + \frac{\left| \frac{-\bar{R}(R_y)}{|\bar{R}|\,|\bar{R}|} + 1 \right|}{|\bar{r} - \bar{R}|} \right]}{\sqrt{1 - \left[\frac{\left| \frac{-\bar{R}(\bar{r} \cdot \bar{R})}{|\bar{R}|\,|\bar{R}|} + \bar{r} \right|}{|\bar{r} - \bar{R}|} \right]^2}} \tag{B-8}$$

$$\frac{\partial el}{\partial z} = \frac{\left[\frac{(-1) \left(\left| \frac{-\bar{R}(\bar{r} \cdot \bar{R})}{|\bar{R}|\,|\bar{R}|} + \bar{r} \right| \right)}{\left(|\bar{r} - \bar{R}| \right)^2} + \frac{\left| \frac{-\bar{R}(R_z)}{|\bar{R}|\,|\bar{R}|} + 1 \right|}{|\bar{r} - \bar{R}|} \right]}{\sqrt{1 - \left[\frac{\left| - \frac{\bar{R}(\bar{r} \cdot \bar{R})}{|\bar{R}|\,|\bar{R}|} + \bar{r} \right|}{|\bar{r} - \bar{R}|} \right]^2}} \tag{B-9}$$

<u>Partial derivatives of azimuth</u>:

$$az = \cos^{-1} \left[\frac{\left(\frac{-\bar{R}(\bar{r} \cdot \bar{R})}{|\bar{R}|\,|\bar{R}|} + \bar{r} \right) \cdot \hat{k}}{\left| \frac{-\bar{R}(\bar{r} \cdot \bar{R})}{|\bar{R}|\,|\bar{R}|} + \bar{r} \right|} \right] \tag{B-10}$$

$$\frac{d}{dx}\left(\cos^{-1}u\right) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \tag{B-11}$$

From eqs (B-10) and (B-11)

$$u = \left[\frac{\left(\frac{-\bar{R}(\bar{r}\cdot\bar{R})}{|\bar{R}|\,|\bar{R}|} + \bar{r}\right)\cdot\hat{k}}{\left|\frac{-\bar{R}(\bar{r}\cdot\bar{R})}{|\bar{R}|\,|\bar{R}|} + \bar{r}\right|}\right] \tag{B-12}$$

the numerator is:

$$\frac{-R_z(xR_x+yR_y+zR_z)}{R_x^2+R_y^2+R_z^2} + z$$

Let: $DOT = xR_x+yR_y+zR_z$

$RSQD = R_x^2+R_y^2+R_z^2$

So the numerator can be written: $z - \frac{R_z\,DOT}{RSQD}$

$$u^2 = \frac{\left(z - \frac{R_zDOT}{RSQD}\right)^2}{\left(x-\frac{R_xDOT}{RSQD}\right)^2 + \left(y-\frac{R_yDOT}{RSQD}\right)^2 + \left(z-\frac{R_zDOT}{RSQD}\right)^2} \tag{B-13}$$

$$\begin{aligned}\frac{du}{dx} = {} & \left(z-\frac{R_zDOT}{RSQD}\right)\cdot(-\tfrac{1}{2})\cdot\left[\left(x-\frac{R_xDOT}{RSQD}\right)^2 + \left(y-\frac{R_yDOT}{RSQD}\right)^2\right.\\ & \left. + \left(z-\frac{R_zDOT}{RSQD}\right)^2\right]^{-3/2} \cdot \left[(2)\cdot\left(x-\frac{R_xDOT}{RSQD}\right)\cdot\left(1-\frac{R_x^2}{RSQD}\right)\right.\\ & \left. + (2)\cdot\left(y-\frac{R_yDOT}{RSQD}\right)\cdot\left(\frac{-R_xR_y}{RSQD}\right) + (2)\cdot\left(z-\frac{R_xDOT}{RSQD}\right)\cdot\left(\frac{-R_xR_z}{RSQD}\right)\right]\\ & + \frac{\frac{-R_xR_z}{RSQD}}{\sqrt{\left(x-\frac{R_xDOT}{RSQD}\right)^2 + \left(y-\frac{R_yDOT}{RSQD}\right)^2 + \left(z-\frac{R_zDOT}{RSQD}\right)^2}}\end{aligned} \tag{B-14}$$

$$\text{so } \frac{d}{dx}(\cos^{-1}u) = \frac{-1}{\sqrt{1 - \dfrac{\left(z - \frac{R_z DOT}{RSQD}\right)^2}{\left(x - \frac{R_x DOT}{RSQD}\right)^2 + \left(y - \frac{R_y DOT}{RSQD}\right)^2 + \left(z - \frac{R_z DOT}{RSQD}\right)^2}}}$$

$$\cdot \left[\frac{(-1)\cdot\left(z - \frac{R_z DOT}{RSQD}\right)\cdot\left[\left(x - \frac{R_x DOT}{RSQD}\right)\cdot\left(1 - \frac{R_x^2}{RSQD}\right) + \left(y - \frac{R_y DOT}{RSQD}\right)\cdot\left(\frac{-R_x R_y}{RSQD}\right) + \left(z - \frac{R_x DOT}{RSQD}\right)\cdot\left(\frac{-R_x R_z}{RSQD}\right.\right.}{\left[\left(x - \frac{R_x DOT}{RSQD}\right)^2 + \left(y - \frac{R_y DOT}{RSQD}\right)^2 + \left(z - \frac{R_x DOT}{RSQD}\right)^2\right]^{3/2}}\right.$$

$$\left. + \frac{\frac{-R_x R_z}{RSQD}}{\sqrt{\left(x - \frac{R_x DOT}{RSQD}\right)^2 + \left(y - \frac{R_y DOT}{RSQD}\right)^2 + \left(z - \frac{R_z DOT}{RSQD}\right)^2}} \right] \qquad \text{(B-15)}$$

$$\frac{du}{dy} = \left(z - \frac{R_z DOT}{RSQD}\right) \cdot -\tfrac{1}{2} \cdot \left[\left(x - \frac{R_x DOT}{RSQD}\right)^2 + \left(y - \frac{R_y DOT}{RSQD}\right)^2 + \left(z - \frac{R_z DOT}{RSQD}\right)^2\right]^{-3/2}$$

$$\cdot \left[(2)\cdot\left(x - \frac{R_x DOT}{RSQD}\right)\cdot\left(\frac{-R_x R_y}{RSQD}\right) + (2)\cdot\left(y - \frac{R_y DOT}{RSQD}\right)^2\cdot\left(1 - \frac{R_y^2}{RSQD}\right) + (2)\cdot\left(z - \frac{R_z DOT}{RSQD}\right)\cdot\left(\frac{-R_y R_z}{RSQ}\right.\right.$$

$$+ \frac{\frac{-R_y R_z}{RSQD}}{\sqrt{\left(x - \frac{R_x DOT}{RSQD}\right)^2 + \left(y - \frac{R_y DOT}{RSQD}\right)^2 + \left(z - \frac{R_z DOT}{RSQD}\right)^2}} \qquad \text{(B-16)}$$

$$\text{so } \frac{d}{dy}\left(\cos^{-1}u\right) = \frac{-1}{\sqrt{1 - \dfrac{\left(z - \dfrac{R_z DOT}{RSQD}\right)^2}{\left(x - \dfrac{R_x DOT}{RSQD}\right)^2 + \left(y - \dfrac{R_y DOT}{RSQD}\right)^2 + \left(z - \dfrac{R_z DOT}{RSQD}\right)^2}}}$$

$$\cdot \left[\frac{(-1)\cdot\left(z - \dfrac{R_z DOT}{RSQD}\right)\left[\left(x - \dfrac{R_x DOT}{RSQD}\right)\cdot\left(\dfrac{-R_x R_y}{RSQD}\right) + \left(y - \dfrac{R_y DOT}{RSQD}\right)\cdot\left(1 - \dfrac{R_y^2}{RSQD}\right) + \left(z - \dfrac{R_z DOT}{RSQD}\right)\cdot\left(\dfrac{-R_y R_z}{RSQD}\right)\right]}{\left[\left(x - \dfrac{R_x DOT}{RSQD}\right)^2 + \left(y - \dfrac{R_y DOT}{RSQD}\right)^2 + \left(z - \dfrac{R_z DOT}{RSQD}\right)^2\right]^{3/2}} \right.$$

$$\left. + \frac{\dfrac{-R_y R_z}{RSQD}}{\left(x - \dfrac{R_x DOT}{RSQD}\right)^2 + \left(y - \dfrac{R_y DOT}{RSQD}\right)^2 + \left(z - \dfrac{R_z DOT}{RSQD}\right)^2} \right] \quad \text{(B-17)}$$

$$\frac{du}{dz} = \left(z - \frac{R_z DOT}{RSQD}\right) \cdot \left[\left(x - \frac{R_x DOT}{RSQD}\right)^2 + \left(y - \frac{R_y DOT}{RSQD}\right) + \left(z - \frac{R_z DOT}{RSQD}\right)\right]^{-3/2}$$

$$\cdot \left[(2) \cdot \left(x - \frac{R_x DOT}{RSQD}\right)\cdot\left(\frac{-R_x R_z}{RSQD}\right) + (2)\cdot\left(y - \frac{R_y DOT}{RSQD}\right)\cdot\left(\frac{-R_y R_x}{RSQD}\right) + (2)\cdot\left(z - \frac{R_z DOT}{RSQD}\right)\cdot\left(1 - \frac{R_z^2}{RSQD}\right) \right.$$

$$+ \frac{\left(1 - \dfrac{R_z^2}{RSQD}\right)}{\sqrt{\left(x - \dfrac{R_x DOT}{RSQD}\right)^2 + \left(y - \dfrac{R_y DOT}{RSQD}\right)^2 \left(\dfrac{R_z DOT}{RSQD}\right)^2}} \quad \text{(B-18)}$$

$$\text{so: } \frac{d}{dz}(\cos^{-1}u) = \frac{-1}{\sqrt{1-\dfrac{\left(z-\dfrac{R_z DOT}{RSQD}\right)^2}{\left(x-\dfrac{R_x DOT}{RSQD}\right)^2+\left(y-\dfrac{R_y DOT}{RSQD}\right)^2+\left(z-\dfrac{R_z DOT}{RSQD}\right)^2}}}$$

$$\cdot\left[\frac{(-1)\cdot\left(z-\dfrac{R_z DOT}{RSQD}\right)\left[\left(x-\dfrac{R_x DOT}{RSQD}\right)\cdot\left(\dfrac{-R_x R_z}{RSQD}\right)+\left(y-\dfrac{R_y DOT}{RSQD}\right)\cdot\left(\dfrac{-R_y R_z}{RSQD}\right)+\left(z-\dfrac{R_z DOT}{RSQD}\right)\cdot\left(1-\dfrac{R_z^2}{RSQD}\right)\right]}{\left[\left(x-\dfrac{R_x DOT}{RSQD}\right)^2+\left(y-\dfrac{R_y DOT}{RSQD}\right)^2+\left(z-\dfrac{R_z DOT}{RSQD}\right)^2\right]^{3/2}}\right.$$

$$\left.+\frac{\left(1-\dfrac{R_z^2}{RSQD}\right)}{\sqrt{\left(x-\dfrac{R_x DOT}{RSQD}\right)^2+\left(y-\dfrac{R_y DOT}{RSQD}\right)^2+\left(z-\dfrac{R_z DOT}{RSQD}\right)^2}}\right] \qquad \text{(B-19)}$$

APPENDIX C

Description of Preliminary Truth Model for Data Generation

The preliminary truth model used in testing the filters was admittedly simple. Variable effects on the trajectory caused by the atmosphere (density changes and winds) and variations in the thrust were neglected. These variations were never added due to the level of performance attained by the filters.

The launch trajectory was elliptical and was generated using modified two-body dynamics. During the initial portion of the trajectory, the effect of thrust was added to the equations of motion in the form of an acceleration. The seven-element truth model state vector consisted of three states for position x, y, and z; three states for velocity - v_x, v_y, and v_z; and a seventh state for acceleration due to thrust - a.

To model acceleration, it was assumed that thrust was constant for each stage and that the initial thrust to weight ratio was known (ref 4:369). The exhaust exit velocity can be determined from

$$V_e = I_{sp}\, g \qquad \text{(C-1)}$$

where V_e = exit velocity of rocket engine exhaust (ft/sec)

I_{sp} = specific impulse of rocket engine (sec)

g = gravity (ft/sec^2)

If the value for thrust is known, the mass flow rate can be obtained from

$$T = V_e \beta \tag{C-2}$$

where T = thrust (lb)

β = mass flow rate (lb-sec/ft)

The acceleration due to thrust is then given as a ratio of thrust to the instantaneous weight (the weight is decreasing as propellant is burned), therefore

$$a = \frac{T}{(m_0 - \beta t)g} = \frac{V_e \beta}{(m_0 - \beta t)g} \tag{C-3}$$

where a = acceleration due to thrust

m_0 = initial mass of the rocket stage and its payload (slug)

t = time (sec)

g = gravity (ft/sec^2)

This can also be expressed as

$$a = \frac{V_e\ M}{(1 - Mt)\ g} \tag{C-4}$$

where $M = \frac{\beta}{M_0}$, a relative mass flow rate

Taking the time derivative gives

$$\dot{a} = \frac{V_e M^2}{(1 - Mt)^2} \tag{C-5}$$

The trajectory resulted from integrating the modified two-body equations of motion:

$$\overline{F} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \\ \dot{a} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ -x/R^3 + (a)(v_x)/v \\ -y/R^3 + (a)(v_y)/v \\ -z/R^3 + (a)(v_z)/v \\ V_e M^2/(1-Mt)^2 \end{bmatrix} \tag{C-6}$$

where

$R^3 = (x^2 + y^2 + z^2)^{3/2}$

$v = (v_x^2 + v_y^2 + v_z^2)^{1/2}$

V_e = exit velocity of propellant from rocket engine

M = relative mass flow rate, the mass flow rate of propellant from the engine divided by the total initial mass of the launch vehicle, β/m_0

t = time

a = acceleration due to thrust; assumed to act in the direction of the velocity vector

Initial conditions on many of the states were critical. Criteria for the position states were:

(1) that the square root of the sum of the squares of the position components equal 1 (distance unit). In other words, the launch point was on the earth's surface.

(2) Coordinates were restricted from any of the axes in order to keep away from special cases.

(3) The launch point should be in the field of view of both observer satellites.

The criteria on the velocities were most critical:

(1) The magnitudes had to be small to simulate near-zero velocity at lift-off, but couldn't be zero because of the need to provide a direction for acceleration.

(2) The relative magnitudes between velocity components needed to follow closely magnitudes between position components for a near-vertical takeoff.

(3) The velocity had to deviate slightly from vertical to result in a gravity turn within the field of view of the observers.

Trial and error was used to come up with proper initial velocity conditions. Velocity magnitudes approximately 10^{-3} that of the position vector proved successful. A velocity magnitude decrease of approximately one percent from vertical was used to result in a gravity turn in the direction of decreased magnitude.

Values for the thrust profile were derived from parameters for a Titan IIIB rocket (ref 6:5-141).

<u>1st stage</u> V_e (exit velocity) - 8243.2 ft/sec

Thrust - 464,900 lb

Propellant mass flow rate - 56.398 lb-sec/ft

Initial mass - 11,275 lb-sec^2/ft

2nd stage V_e - 10,220.3 ft/sec

Thrust - 102,300 lb

Propellant mass flow rate - 10.0095 lb-sec/ft

Initial mass - 2735 lb-sec^2/ft

3rd stage V_e - 9402.4 ft/sec

Thrust - 16,000 lb

Propellant mass flow rate - 1.7017 lb-sec/ft

Initial mass - 455 lb-sec^2/ft

To simulate staging, time dependent conditional commands were used to select a different subroutine for each stage. At the beginning of each stage the acceleration would begin with a nominal value, V_eM. Acceleration increased, reaching its maximum just before staging as shown in Figure C-1.

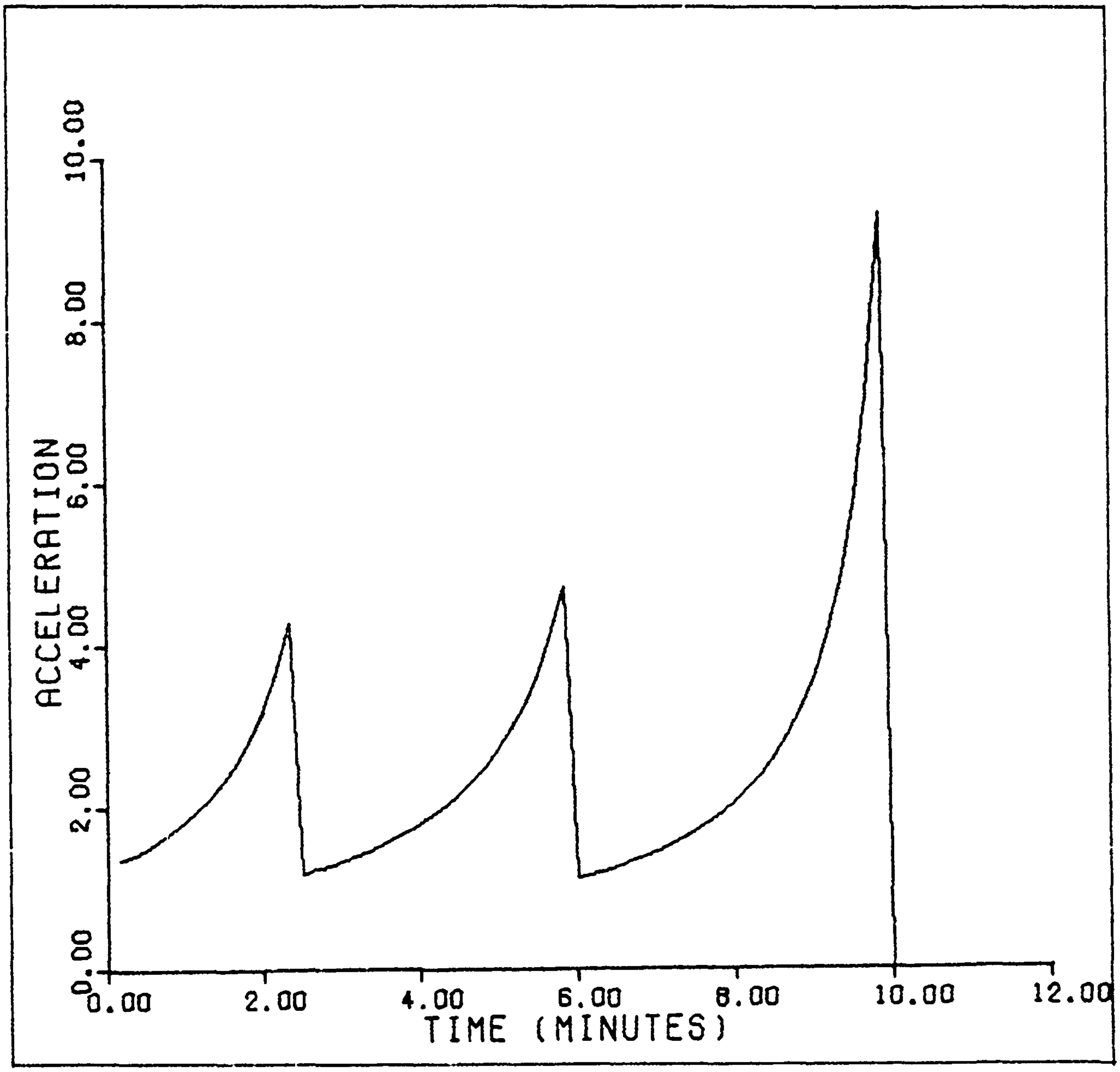

Figure C-1. Truth model acceleration versus time

VITA

Gregory D. Miller [redacted]. He graduated from high school in Grand Junction, Colorado in 1972 and entered the United States Air Force Academy from which he received a Bachelor of Science Degree in Engineering Science in 1976. After graduation he worked as a test engineer and a program engineer in the 3246th Test Wing, Eglin AFB, Florida. He attended Squadron Officer's School from April to June, 1980, after which he entered the School of Engineering, Air Force Institute of Technology.

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FREDRIC C. LYNCH, Major, USAF

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Estimation
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parameter identification

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A seven-state inverse covariance (Bayes) filter was implemented to determine performance parameters of a launch vehicle. Data measurements were restricted to azimuth and elevation readings typical of data from an infrared sensor in geosynchronous orbit. Results of this study indicate that the magnitude of constant acceleration, assumed to act in the direction of velocity, can be estimated using a seven-state filter (3 states each for position and velocity, and a seventh state for acceleration). The system

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) is unobservable for short arcs if only one observer is available. The addition of a second observer can allow the system to be observed. An ad hoc fading - memory technique, in which confidence in the seventh state estimate was decreased, proved unsuccessful in estimating variable acceleration of a launch vehicle. Further attempts at estimating variable acceleration with an eight-state filter (3 states each for position and velocity, and seventh and eighth states involving engine exit velocity and propellent mass flow rate) were unsuccessful.